Unit 4 Review

Evaluate the limits below. L'Hopital's rule is not necessarily required. Show EVERY STEP.

23. $\lim\limits_{x\to 0} \dfrac{x + \sin 3x}{x - \sin 3x}$

24. $\lim\limits_{x\to 0} \dfrac{e^{4x} - 1}{\cos x}$

25. $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$

26. $\lim\limits_{x\to 0} \dfrac{x + \tan 2x}{x - \tan 2x}$

~~27.~~ $\lim\limits_{x\to 0} \dfrac{\tan 2x}{\tanh 3x}$

28. $\lim\limits_{x\to 0} \dfrac{2x - \sin^{-1} x}{2x + \cos^{-1} x}$

29. $\lim\limits_{x\to 0} \dfrac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$

30. $\lim\limits_{x\to -\infty} xe^{x}$

23. $\lim\limits_{x\to 0} \dfrac{x+\sin(3x)}{x-\sin(3x)} \overset{H}{=} \lim\limits_{x\to 0} \dfrac{1+3\cos(3x)}{1-3\cos(3x)} = \dfrac{4}{-2} = -2$

24. $\lim\limits_{x\to 0} \dfrac{e^{4x}-1}{\cos(x)} = \dfrac{0}{1} = 0$

Chain rule

25. $\lim\limits_{x\to 0} \dfrac{\tan(x)-\sin(x)}{x^3} \overset{H}{=} \lim\limits_{x\to 0} \dfrac{\sec^2(x)-\cos(x)}{3x^2} \overset{H}{=} \lim\limits_{x\to 0} \dfrac{2\sec^2(x)\tan(x)+\sin(x)}{6x} \overset{H}{=} \lim\limits_{x\to 0} \dfrac{2\sec^2(x)\sec^2(x)+4\sec^2(x)\tan^2(x)+\cos(x)}{6}$

$= \dfrac{2+0+1}{6} = \dfrac{1}{2}$

26. $\lim\limits_{x\to 0} \dfrac{x+\tan(2x)}{x-\tan(2x)} \overset{H}{=} \lim\limits_{x\to 0} \dfrac{1+\sec^2(2x)\cdot 2}{1-\sec^2(2x)\cdot 2} = \dfrac{1+2}{1-2} = -3$

28. $\lim\limits_{x\to 0} \dfrac{2x-\sin^{-1}(x)}{2x+\cos^{-1}(x)} = \dfrac{0-0}{0+\frac{\pi}{2}} = 0$

29. $\lim\limits_{x\to 0} \dfrac{2x-\sin^{-1}(x)}{2x+\tan^{-1}(x)} \overset{H}{=} \lim\limits_{x\to 0} \dfrac{2-\frac{1}{\sqrt{1-x^2}}}{2+\frac{1}{1+x^2}} = \dfrac{2-1}{2+1} = \dfrac{1}{3}$

Rewrite

30. $\lim\limits_{x\to -\infty} xe^{x} = \lim\limits_{x\to -\infty} \dfrac{x}{e^{-x}} \overset{H}{=} \lim\limits_{x\to -\infty} \dfrac{1}{-e^{-x}} = 0$

Let f be the function defined by $f(x) = 2x + 3e^{-5x}$, and let g be a differentiable function with derivative given by $g'(x) = \frac{1}{x} + 4\cos\left(\frac{5}{x}\right)$. It is known that $\lim_{x\to\infty} g(x) = \infty$. The value of $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \to \frac{\infty}{\infty}$$

L'Hopital's Rule

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \overset{H}{=} \lim_{x\to\infty} \frac{2+3e^{-5x}(-5)}{\frac{1}{x}+4\cos\left(\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{2-\frac{15}{e^{5x}}}{\frac{1}{x}+4\cos\left(\frac{5}{x}\right)} = \frac{2-0}{0+4\cos(0)} = \frac{1}{2}$$

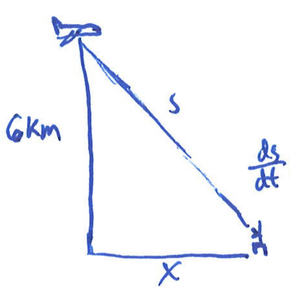
Let f be the function defined by $f(x) = 3x + 2e^{-3x}$, and let g be a differentiable function with derivative given by $g'(x) = 4 + \frac{1}{x}$. It is known that $\lim_{x\to\infty} g(x) = \infty$. The value of $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \Rightarrow \frac{\infty}{\infty}$$

L'Hopital's rule

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \overset{H}{=} \lim_{x\to\infty} \frac{3-6e^{-3x}}{4+\frac{1}{x}} = \lim_{x\to\infty} \frac{3-\frac{6}{e^{3x}}}{4+\frac{1}{x}} = \frac{3}{4}$$

1. An airplane is flying towards a radar station at a constant height of 6 km above the ground. If the distance s between the airplane and the radar station is decreasing at a rate of 400 km per hour when s = 10 km., what is the horizontal speed of the plane?



$\frac{ds}{dt} = -400$ km/hr when $s = 10$ km, what is $\frac{dx}{dt}$?

$$x^2 + 6^2 = s^2$$
$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$
$$x\frac{dx}{dt} = s\frac{ds}{dt}$$
$$\frac{dx}{dt} = \frac{s\frac{ds}{dt}}{x}$$

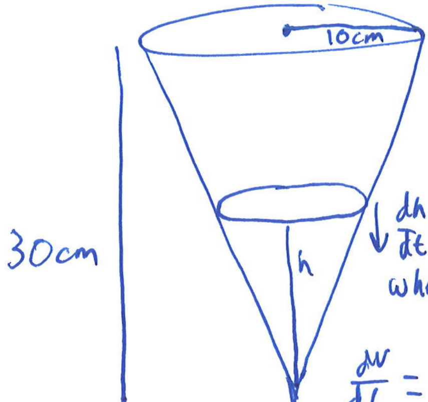
$$x^2 + 6^2 = 10^2$$
$$x^2 + 36 = 100$$
$$x^2 = 64$$
$$x = 8$$

$$\frac{dx}{dt} = \frac{10(-400)}{8} = \frac{-4000}{8} = -500 \text{ Km/hr}$$

speed is $|-500|$

$\boxed{\text{Speed: } 500 \text{ km/hr}}$

6. A funnel in the shape of an inverted cone is 30 cm deep and has a diameter across the top of 20 cm. Liquid is flowing out of the funnel at the rate of 12 cm^3/sec. At what rate is the height of the liquid decreasing at the instant when the liquid in the funnel is 20 cm deep?



$\frac{dh}{dt} = ?$ when $h = 20$cm

$\frac{dV}{dt} = -12\text{cm}^3/\text{sec}$

$$V = \frac{1}{3}\pi r^2 h$$
$$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$$
$$V = \frac{1}{3}\pi\frac{h^3}{9}$$
$$V = \frac{1}{27}\pi h^3$$
$$\frac{dV}{dt} = \frac{1}{9}\pi h^2\frac{dh}{dt}$$

$$\frac{r}{h} = \frac{10}{30}$$
$$\frac{r}{h} = \frac{1}{3}$$
$$r = \frac{h}{3}$$

$$-12 = \frac{1}{9}\pi(20)^2\frac{dh}{dt}$$
$$\frac{-12\cdot 9}{\pi(400)} = \frac{dh}{dt}$$
$$\boxed{\frac{-27}{100\pi} \text{ cm/sec}} = \frac{dh}{dt}$$

In #1-5, answer the following questions for each position function s(t) in meters where t is in seconds if a particle is moving along the x-axis. Assume you are only concened with the given intervals

a) What is the velocity function?
   What is the velocity at t = 3 seconds?
b) When is the particle at rest?
c) When is the particle moving right? Moving left?
d) What is the acceleration function?
   What is the acceleration at t = 1 second?
e) What is the displacement and total distance traveled for the indicated interval specific to each problem?
f) When is the particle speeding up? Slowing down?
g) Find the velocity when the acceleration is 0.

1. $s(t) = t^3 - 3t + 3$ displacement and total distance traveled in [0, 6]

2. $s(t) = t^3 - 6t^2$ displacement and total distance traveled in [0, 7]

3. $s(t) = 2t^3 - 21t^2 + 60t + 3$ displacement and total distance traveled in [0, 8]

4. $s(t) = 2t^3 - 14t^2 + 22t - 5$ displacement and total distance traveled in [0, 6]

5. $s(t) = 2t^3 - 15t^2 + 24t + 8$ displacement and total distance traveled in [0, 5]

Showing some answers in full: summary of answers on the next page.

1. $s(t) = t^3 - 3t + 3$

a) $\boxed{v(t) = 3t^2 - 3}$
$\boxed{v(3) = 24 \text{ m/s}}$

b) $v(t) = 0$
$3t^2 - 3 = 0$
$3t^2 = 3$
$t^2 = 1$
$t = \pm 1$ but assume $0 \le t \le 6$
$\boxed{t = 1s}$

c) Moving Right when $v(t)$ is +; left when $v(t)$ is −
$v(t) = 3t^2 - 3 = 3(t^2 - 1) = 3(t-1)(t+1)$

	⊕	−1 ⊖	1 ⊕
3	+	+	+
t−1	−	−	+
t+1	−	+	+

Particle moves right (1, 6]
Particle moves left (0, 1)

d) $\boxed{a(t) = 6t}$
$\boxed{a(1) = 6 \text{ m/s}^2}$

	⊖	0 ⊕
6t	−	+

e) displacement is change in position:
$s(6) - s(0) = 201 - 3 = \boxed{198 m}$
Total distance traveled would take into account turn around spots
$s(0) = 3 \xrightarrow{2m} s(1) = 1 \quad s(6) = 201$ (200 m)
Total distance: $\boxed{202m}$

f)

		−1		0		1	
a(t)	−		−		+		+
v(t)	+		−		−		+

Particle is speeding up on (1, 6] b/c v(t) and a(t) have the same sign
Particle is slowing down on (0, 1) b/c v(t) and a(t) have different signs.

g) $a(t) = 0$
$6t = 0$
$t = 0$
$v(0) = \boxed{-3 m/s}$

2. $s(t) = t^3 - 6t^2$

a) $\boxed{v(t) = 3t^2 - 12t}$
$v(3) = 27 - 36 = \boxed{-9 m/s}$

b) $v(t) = 0 \quad 3t^2 - 12t = 0$
$3t(t-4) = 0$
$\boxed{t = 0s,\ t = 4s}$

c)

	⊕	0 ⊖	4 ⊕
3t	−	+	+
t−4	−	−	+

Particle is moving right (4, 7]
Particle is moving left (0, 4)

d) $\boxed{a(t) = 6t - 12}$
$\boxed{a(1) = -6 \text{ m/s}^2}$

	⊖	2 ⊕
6	+	+
t−2	−	+

e) displacement:
$s(7) - s(0) = 49 - 0 = \boxed{49m}$
Total distance traveled
$s(0) = 0 \xrightarrow{32} s(4) = -32 \xrightarrow{81} s(7) = 49$
Total distance traveled: $\boxed{113 m}$

f)

		0		2		4	
a(t)	−		−		+		+
v(t)	+		−		−		+

Particle is speeding up on (0, 2), (4, 7]
Particle is slowing down on (2, 4)

g) $a(t) = 0$ at $t = 2$ $\quad v(2) = \boxed{-12 m/s}$

Answers to Particle Motion Problems

#1) a) $v(t) = 3t^2-3$ $v(3)= 24$ m/s b) $t = 1$ sec c) moving rt (1, 6) sec Left (0, 1) sec d) $a(t) = 6t$ sec $a(1) = 6\ {}^{m}\!/_{\sec^2}$ e) disp 198 m tdt: 202 m f) speeding up (1, 6) sec slowing down (0, 1) sec g) $a(t) = 0$ when $t = 0$. $v(0) = -3$ m/s

#2) a) $v(t) = 3t^2-12t$ $v(3) = -9$ m/sec b) $t = 0, 4$ sec c) moving rt (4, 7) sec, left (0, 4) sc (d) $a(t) = 6t-12$ $a(1) = -6\ {}^{m}\!/_{\sec^2}$ (e) disp 49 m dt: 113 m f) speeding up (0, 2), (4, 7) s slowing down (2, 4) g) $a(t)=0$ when $t = 2$ sec $v(2) = -12$ m/sec

#3) a) $v(t) = 6t^2-42t+60$ $v(3) = -12$ m/s (b) $t = 2, 5$ sec (c) right (0, 2) (5, 8) sec left: (2, 5) sec (d) $a(t) = 12t - 42$ $a(1) = -30\ {}^{m}\!/_{\sec^2}$ (e) disp 160 m tdt: 214 m f) speeding up (2, 3.5) (5, 8) sec slowing down (0, 2) sec (3.5, 5) sec
(g) $t = 3.5$ $v(3.5) = -13.5$ m/sec

#4) a) $v(t) = 6t^2-28t+22$ $v(3) = -8$ m/s (b) $t = 1, \frac{11}{3}$ sec (c) moving rt (0, 1) $(\frac{11}{3}, 6)$ s moving left $(1, \frac{11}{3})$ sec (d) $a(t) = 12t - 28$ $a(1) = -16$ m/sec^2 (e) disp 60m, tdt 97.9258 m

(f) speeding up $(1, \frac{7}{3})$ $(\frac{11}{3}, 6)$ sec slowing down (0, 1) $(\frac{7}{3}, \frac{11}{3})$sec

(g) $v(\frac{7}{3}) = -10.66667$ m/sec

#5) a) $v(t) = 6t^2-30t+24$ $v(3)= -12$ m/s (b) $t = 1, 4$ sec (c) right (0, 1) (4, 5) sec left (1, 4) sec (d) $a(t) = 12t - 30$ $a(1) = -18\ {}^{m}\!/_{\sec^2}$ (e) disp 5 m tdt 49 m
(f) speeding up (1, 2.5) (4, 5) sec slowing down (0, 1) (2.5, 4) sec
(g) $t = 2.5$ $v(2.5) = -13.5$ m/sec

6. The function $K(t)$ measures the rate of change of population of a kangaroos at the Karlamilyi National Park in Western Australia in kangaroos per year where t is in years since 1990. What does K'(2) mean in the context of this situation?

K'(2) is the rate of change of the Kangaroo population rate of change in Kangaroos/yr^2 at $t=2$yrs(1992)
It will tell us whether the rate of change of kangaroo population is increasing or decreasing.

7. a. The function $V(t)$ measures the volume of water in a pool in gallons where t is in minutes since the pool began to be filled. What does V'(32) mean in the context of this situation?

V'(32) is rate of change of volume of water in the pool in gal/minute at $t=32$ minutes (32 minutes after the pool begins to be filled. It tells us how the quickly the pool is being filled or drained.

b. If V'(7) is negative, what does that mean for the volume of water in the pool?

If V'(7) is negative, that means the volume of water in the pool is decreasing at $t=7$ minutes. The pool is draining.

8. For the function f, $f'(x) = \ln(x) + \frac{1}{2}$ and $f(e^{1/2})=2$. What is the approximation for f(1.7) found by using the line tangent to the graph of f at $x=e^{1/2}$? Select your answer below.

Point: $(e^{1/2}, 2)$

Slope: $f'(e^{1/2}) = \ln(e^{1/2}) + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$

$y-2 = 1(x-e^{1/2})$

$y = 1(x-e^{1/2})+2 \longrightarrow$ at $x=1.7$ $\boxed{y=1(1.7-e^{1/2})+2}$ Leave this! $\rightarrow f(1.7) \approx 2.051$ Decimal answer

9. Find the linearization of $f(x) = (3x^2 - 12)^2$ at x=1. Then use your linearization to approximate f(0.9).

Point: $f(1) = (3-12)^2 = 81$

$(1, 81)$

Slope: $f'(x) = 2(3x^2-12)\cdot 6x$

$f'(1) = 2(-9)\cdot 6 = -108$

$y-81 = -108(x-1)$

$y = -108(x-1)+81$

$\boxed{y = -108(0.9-1)+81}$ Leave this!

$y = -108(-0.1)+81 \rightarrow f(0.9) \approx 91.8$

10. For the function f, $f'(3) = -1$ and f(3)=11. What is the approximation for f(2.9) found by using the line tangent to the graph of f at x=3?

$y-11 = -1(x-3)$

$y = -1(x-3)+11$

$\boxed{y = -1(2.9-3)+11}$ leave this!

$\hookrightarrow f(2.9) \approx -1(-0.1)+11 = 11.1$

11. CALC OK

a. If $v(t) = \frac{1}{2}e^{\sin\left(\frac{t}{4}\right)} - 7\sin\left(\frac{\cos(t)}{4}\right)$ models the velocity of a particle for 0<t<10 as the particle moves along the x-axis, at what time(s) is the particle at rest? Justify your answer.

The particle is at rest when $v(t)=0$: $t=1.178$, 5.591, and 6.975.

b. On what interval(s) is is the particle moving to the right? Justify your answer.

The particle is moving right when $v(t)$ is positive: $(1.178, 5.591)$ and $(6.975, 10]$

c. On what intervals is the acceleration of the particle positive? Justify your answer.

Acceleration is positive when $v(t)$ is increasing ($v(t)$ has a positive slope): $[0, 3.247)$ and $(2\pi, 9.320)$

d. Is the particle speeding up or slowing down at t=3 Justify your answer.

$v(3) = 2.703$

$a(3) = 0.420$ The particle is speeding up because $v(3)$ and $a(3)$ have the same sign.

↳ use derivative function on calculator math → 8:nDeriv

e. What is the acceleration of the particle the first time that velocity is 0? Show the work that leads to your answer.

$v(t)=0$ for the first time at $t=1.1782503$ ← Don't round middle steps!

$a(1.1782503) = 1.769$

f. Another particle is traveling at a velocity of b(t)=-1. Are there any times when both particles have the same velocity? If so, at what time does this occur?

Yes, at $t=0.453$, both particles have a velocity of -1.

(Find the time of intersection on your calculator).

12. *CHALLENGE:*
Galileo discovered that the height s(t) and velocity v(t) of an object tossed vertically in the air are given as functions of time by the formulas

$$s(t) = s_0 + v_0 t - \frac{1}{2} g t^2, \qquad v(t) = \frac{ds}{dt} = v_0 - gt$$

where s_0 is the position at time t = 0,

v_0 is the velocity at t = 0 and g is the

acceleration due to gravity on the surface ot

the earth with the value g $\approx$ 32 $^{ft}/_{\sec^2}$ *or* 9.8 $^{m}/_{\sec^2}$

A slingshot launches a stone vertically with an initial velocity of 300 ft/s from an initial height of 6 feet.
(a) Find the position of the stone as a function of time t.
(b) Find the velocity as a function of time t.
(c) Find the velocity at t = 2 sec.
(d) What is the stone's maximum height and when does it reach that height?

a) $s(t) = 6 + 300t - \frac{1}{2}(32)t^2 = 6 + 300t - 16t^2$

b) $v(t) = 300 - 32t$

c) $v(2) = 300 - 64 = 236$ ft/s

d) When a projectile which is thrown upwards has a maximum height, its velocity is 0. It stops for a split second before it drops back down.

$300 - 32t = 0$

$300 = 32t$

$\frac{300}{32} = t$

$\boxed{t = 9.375 \text{ sec}}$ $\boxed{s(9.375) = 1412.25 \text{ ft}}$

Extra Practice: Textbook: P. 248 #41-43, P. 258 21-24, P. 458 1-16

Check back of book or talk to Markwalter for answers/help w/ these.