Solving Differential Equations

SWBAT:

Match slope fields to their differential equations & properties.

Solve differential equations by separation of variables and FTC.

Announcement

- Test Friday
  - This is a very short unit.
  - Khan Assigned.
- Office Hours in P3 during lunch this week and next.

Great Opportunity

Wednesday 1/15 and Thursday 1/16

KSJC Theater Club will produce *The Thing* at KIPP Heartwood

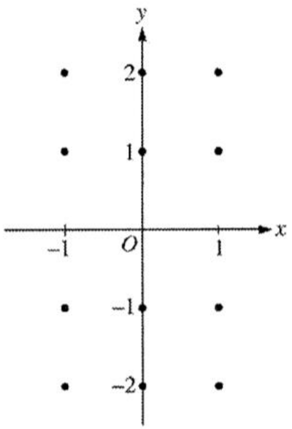
Tickets on sale at lunch in P3

Question #5

Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.

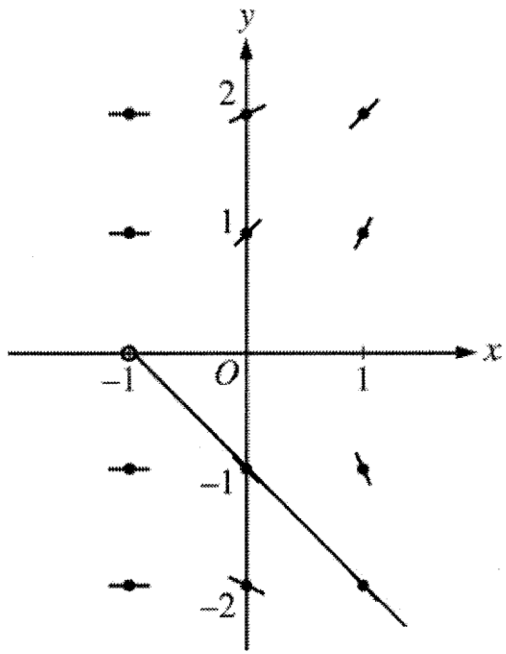
(Note: Use the axes provided in the exam booklet.)



(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane for which $y \neq 0$. Describe all points in the xy-plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.

(a)



$3 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \\ 1 : \text{solution curve through } (0, -1) \end{cases}$

(b) $-1 = \dfrac{x+1}{y} \Rightarrow y = -x - 1$

$\dfrac{dy}{dx} = -1$ for all (x, y) with $y = -x - 1$ and $y \neq 0$

1 : description

(c) $\int y \, dy = \int (x+1) \, dx$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} + x + C$

$\dfrac{(-2)^2}{2} = \dfrac{0^2}{2} + 0 + C \Rightarrow C = 2$

$y^2 = x^2 + 2x + 4$

Since the solution goes through $(0,-2)$, y must be negative. Therefore $y = -\sqrt{x^2 + 2x + 4}$.

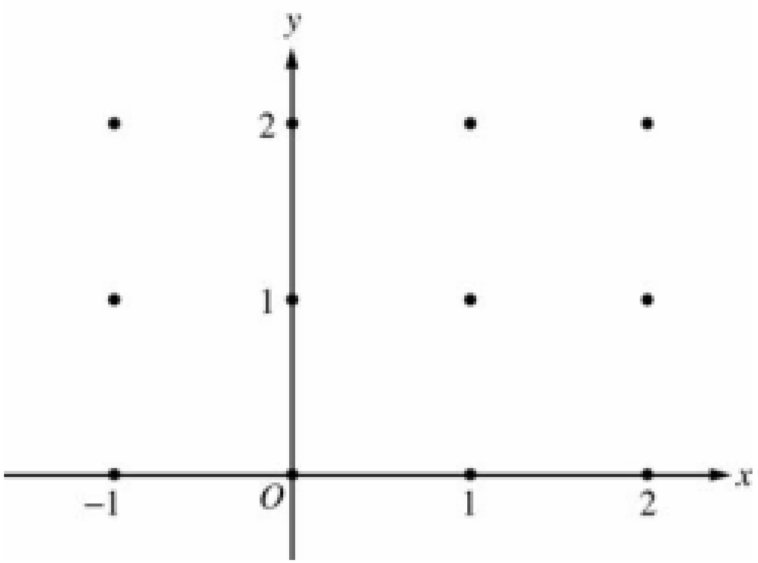
$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

2005 Form B **Question 6**

Consider the differential equation $\frac{dy}{dx} = \frac{-xy^2}{2}$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 2$.

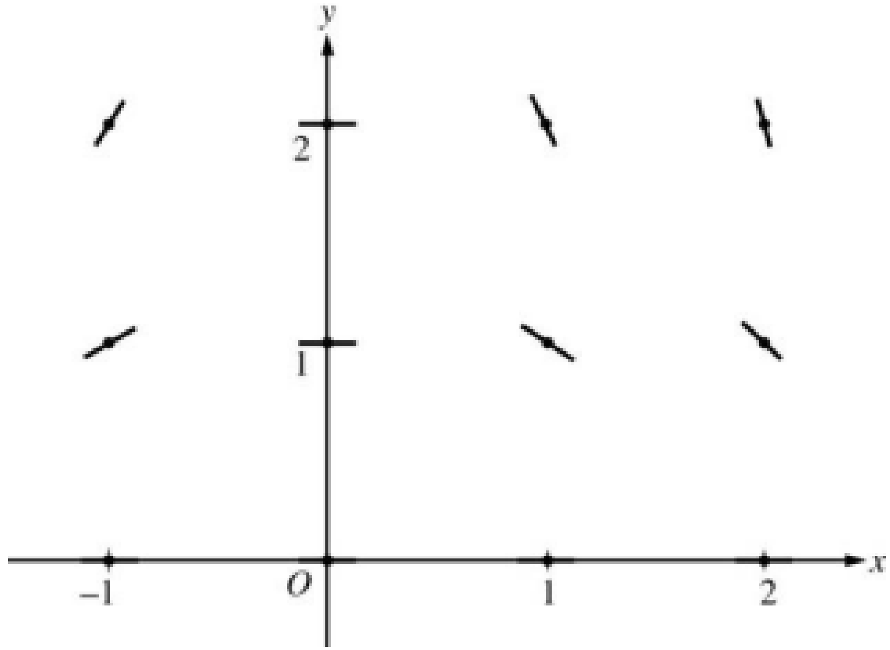


(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the test booklet.)

(b) Write an equation for the line tangent to the graph of f at $x = -1$.

(c) Find the solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

(a)



$2: \begin{cases} 1: \text{zero slopes} \\ 1: \text{nonzero slopes} \end{cases}$

(b) Slope $= \dfrac{-(-1)4}{2} = 2$

$y - 2 = 2(x+1)$

1 : equation

(c) $\dfrac{1}{y^2}dy = -\dfrac{x}{2}dx$

$-\dfrac{1}{y} = -\dfrac{x^2}{4} + C$

$-\dfrac{1}{2} = -\dfrac{1}{4} + C;\ C = -\dfrac{1}{4}$

$y = \dfrac{1}{\dfrac{x^2}{4} + \dfrac{1}{4}} = \dfrac{4}{x^2+1}$

$6: \begin{cases} 1: \text{separates variables} \\ 2: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } y \end{cases}$

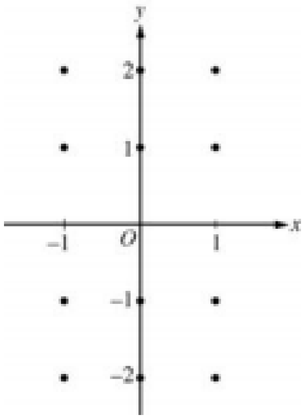
Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

2005 **Question 6**

Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

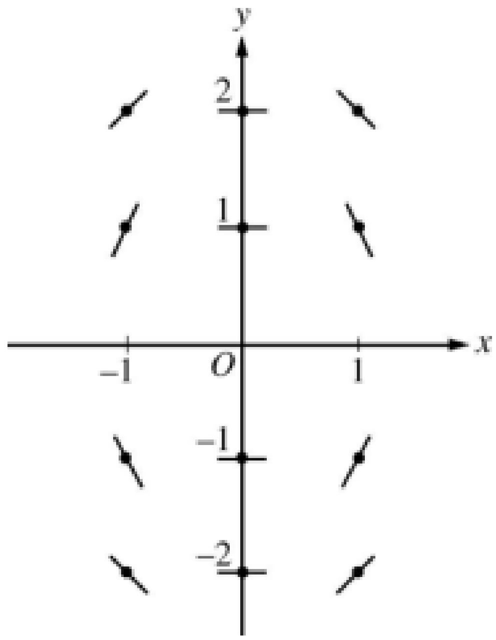
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the pink test booklet.)



(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1, -1)$ and use it to approximate $f(1.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

(a)



$2: \begin{cases} 1: \text{zero slopes} \\ 1: \text{nonzero slopes} \end{cases}$

(b) The line tangent to f at $(1, -1)$ is $y + 1 = 2(x - 1)$. Thus, $f(1.1)$ is approximately -0.8.

$2: \begin{cases} 1: \text{equation of the tangent line} \\ 1: \text{approximation for } f(1.1) \end{cases}$

(c) $\dfrac{dy}{dx} = -\dfrac{2x}{y}$

$y\,dy = -2x\,dx$

$\dfrac{y^2}{2} = -x^2 + C$

$\dfrac{1}{2} = -1 + C;\ C = \dfrac{3}{2}$

$y^2 = -2x^2 + 3$

Since the particular solution goes through $(1, -1)$, y must be negative.

Thus the particular solution is $y = -\sqrt{3 - 2x^2}$.

$5: \begin{cases} 1: \text{separates variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } y \end{cases}$

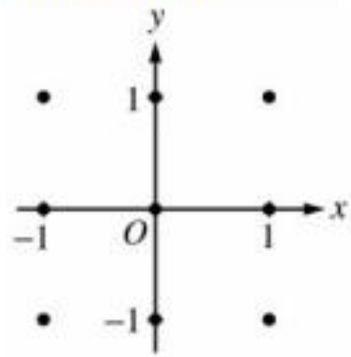
Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

2006 Form B **Question 5**

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

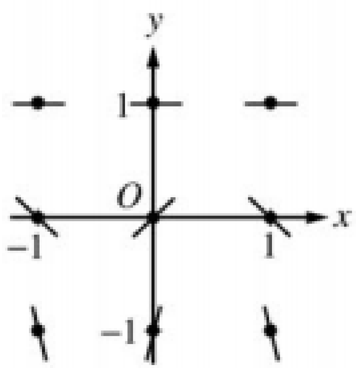
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

(a)



$2: \begin{cases} 1: \text{zero slopes} \\ 1: \text{all other slopes} \end{cases}$

(b) The line $y = 1$ satisfies the differential equation, so $c = 1$.

$1: c = 1$

(c) $\dfrac{1}{(y-1)^2}\,dy = \cos(\pi x)\,dx$

$-(y-1)^{-1} = \dfrac{1}{\pi}\sin(\pi x) + C$

$\dfrac{1}{1-y} = \dfrac{1}{\pi}\sin(\pi x) + C$

$1 = \dfrac{1}{\pi}\sin(\pi) + C = C$

$\dfrac{1}{1-y} = \dfrac{1}{\pi}\sin(\pi x) + 1$

$\dfrac{\pi}{1-y} = \sin(\pi x) + \pi$

$y = 1 - \dfrac{\pi}{\sin(\pi x) + \pi}$ for $-\infty < x < \infty$

$6: \begin{cases} 1: \text{separates variables} \\ 2: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{answer} \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

Solving Differential Equations

If $\frac{dy}{dx} = f'(x)$ and $f(a) = y_0$, then the specific solution to the differential equation is:

If y = *f*(x) is a solution to the differential equation $\frac{dy}{dx} = e^{x^2}$ with the initial condition *f*(0) = 2, which of the following is true?

A) $f(x) = 1 + e^{x^2}$

B) $f(x) = 2xe^{x^2}$

C) $f(x) = \int_1^x e^{t^2}\,dt$

D) $f(x) = 2 + \int_0^x e^{t^2}\,dt$

E) $f(x) = 2 + \int_2^x e^{t^2}\,dt$

If $f'(x) = \frac{2}{x}$ and $f(\sqrt{e}) = 5$ then $f(e) =$

A) 2

B) *ln* 25

C) $5 + \frac{2}{e} - \frac{2}{e^2}$

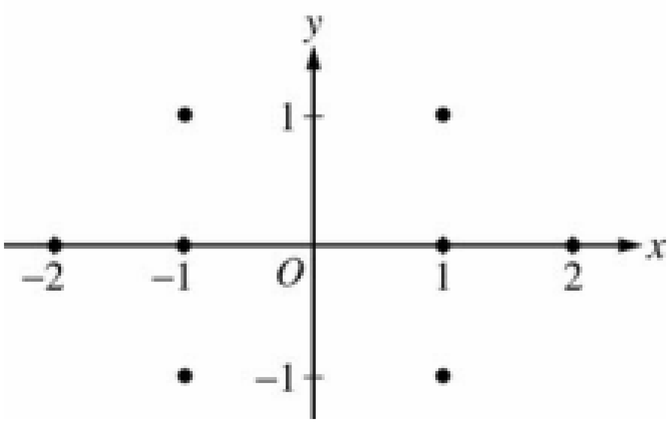
D) 6

E) 25

2006 **Question 5**

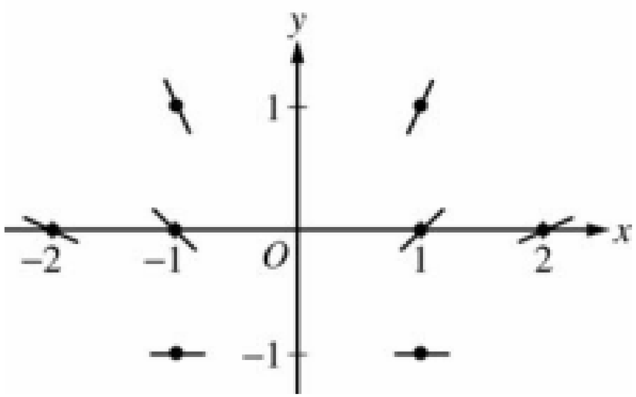
Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated. **(Note: Use the axes provided in the pink exam booklet.)**



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

(a)



2 : sign of slope at each point and relative steepness of slope lines in rows and columns

(b) $\dfrac{1}{1+y}\,dy = \dfrac{1}{x}\,dx$

$\ln|1+y| = \ln|x| + K$

$|1+y| = e^{\ln|x| + K}$

$1 + y = C|x|$

$2 = C$

$1 + y = 2|x|$

$y = 2|x| - 1$ and $x < 0$

or

$y = -2x - 1$ and $x < 0$

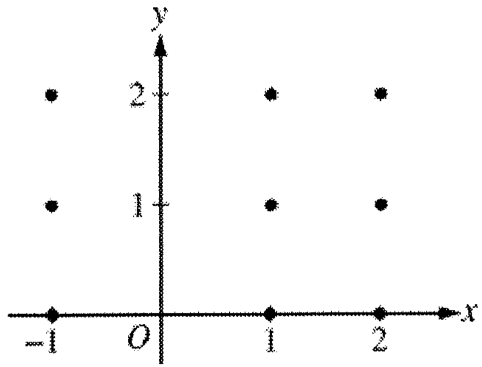
7 :
- 6 :
  - 1 : separates variables
  - 2 : antiderivatives
  - 1 : constant of integration
  - 1 : uses initial condition
  - 1 : solves for y
- Note: max 3/6 [1-2-0-0-0] if no constant of integration
- Note: 0/6 if no separation of variables
- 1 : domain

2008

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

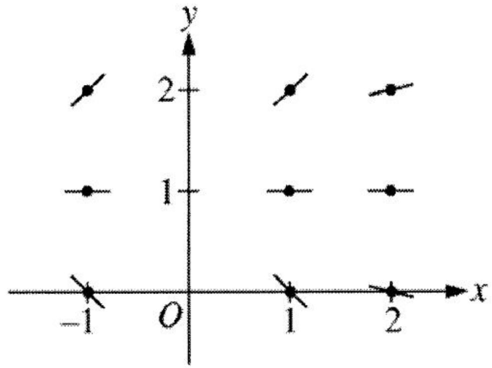
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x \to \infty} f(x)$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) $\frac{1}{y-1}\,dy = \frac{1}{x^2}\,dx$

$\ln|y-1| = -\frac{1}{x} + C$

$|y-1| = e^{-\frac{1}{x}+C}$

$|y-1| = e^{C}e^{-\frac{1}{x}}$

$y - 1 = ke^{-\frac{1}{x}}$, where $k = \pm e^{C}$

$-1 = ke^{-\frac{1}{2}}$

$k = -e^{\frac{1}{2}}$

$f(x) = 1 - e^{\left(\frac{1}{2}-\frac{1}{x}\right)}$, $x > 0$

$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antidifferentiates} \\ 1 : \text{includes constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

(c) $\lim_{x\to\infty} 1 - e^{\left(\frac{1}{2}-\frac{1}{x}\right)} = 1 - \sqrt{e}$

1 : limit

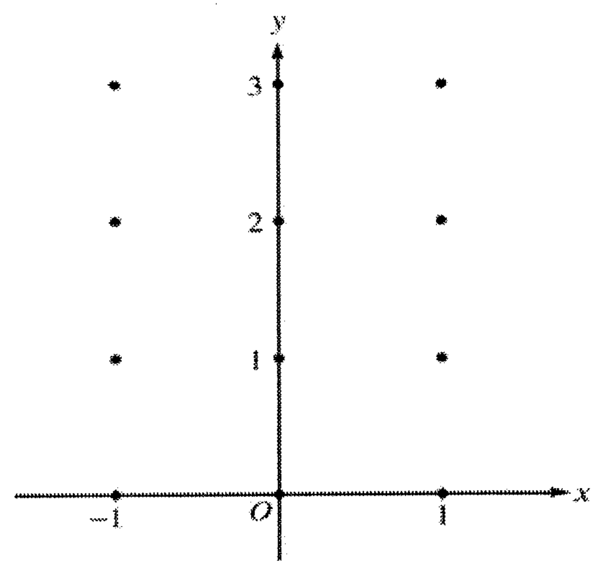
Question 6

Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$.

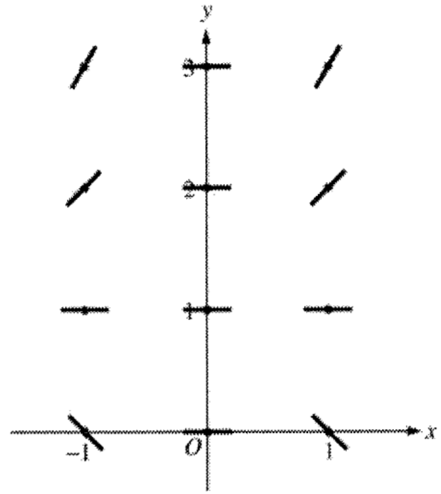
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the pink test booklet.)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the *xy*-plane. Describe all points in the *xy*-plane for which the slopes are positive.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$.



(a)



2 : { 1 : zero slope at each point (x, y) where $x = 0$ or $y = 1$
1 : { positive slope at each point (x, y) where $x \neq 0$ and $y > 1$; negative slope at each point (x, y) where $x \neq 0$ and $y < 1$ }

(b) Slopes are positive at points (x, y) where $x \neq 0$ and $y > 1$.

1 : description

(c)

$$\frac{1}{y-1}dy = x^2 dx$$

$$\ln|y-1| = \frac{1}{3}x^3 + C$$

$$|y-1| = e^C e^{\frac{1}{3}x^3}$$

$$y - 1 = Ke^{\frac{1}{3}x^3},\ K = \pm e^C$$

$$2 = Ke^0 = K$$

$$y = 1 + 2e^{\frac{1}{3}x^3}$$

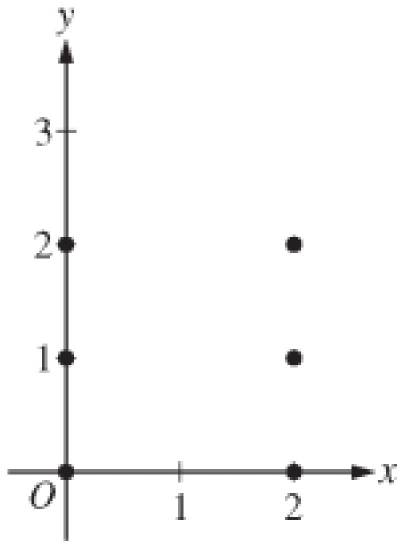
6 : { 1 : separates variables
2 : antiderivatives
1 : constant of integration
1 : uses initial condition
1 : solves for y
0/1 if y is not exponential }

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

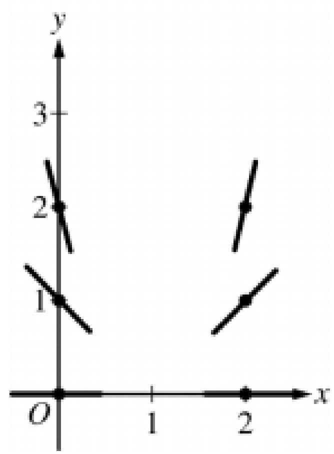
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

(b) $\left.\dfrac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} = \dfrac{3^2}{2-1} = 9$

An equation for the tangent line is $y = 9(x-2)+3$.

$f(2.1) \approx 9(2.1-2)+3 = 3.9$

$2 : \begin{cases} 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$

(c) $\dfrac{1}{y^2}\,dy = \dfrac{1}{x-1}\,dx$

$\displaystyle\int \frac{1}{y^2}\,dy = \int \frac{1}{x-1}\,dx$

$-\dfrac{1}{y} = \ln|x-1| + C$

$-\dfrac{1}{3} = \ln|2-1| + C \Rightarrow C = -\dfrac{1}{3}$

$-\dfrac{1}{y} = \ln|x-1| - \dfrac{1}{3}$

$y = \dfrac{1}{\frac{1}{3} - \ln(x-1)}$

Note: This solution is valid for $1 < x < 1 + e^{1/3}$.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/5$ [1-2-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

1. If $f'(x) = -\sin(x)$ and $f\left(\frac{\pi}{2}\right) = 3$, what is $f\left(\frac{3\pi}{4}\right) =$

Homework: Finish 7.4 Packet