Lesson 7a: Geometric Sequences and Exponential Growth and Decay

Classwork

Think back to the 2nd M&M activity. About what is the value of b? Why?

$b \approx 0.5$

$b = (1 - 50\%) = (1 - 0.5)$

Can the function ever become 0? In other words, can the trials yield 0?

In reality yes! You could roll one M&M and lose it.

Exponential functions change by a common factor/ratio.

Opening Exercise

Suppose a ball is dropped from an initial height h_0 and that each time it rebounds, its new height is 60% of its previous height.

a. What are the first four rebound heights h_1, h_2, h_3, and h_4 after being dropped from a height of $h_0 = 10$ ft.?

$h_0 = 10$ ft

$h_1 = 6$ ft

$h_2 = 3.6$ ft

$h_3 = 2.16$ ft

b. Suppose the initial height is A ft. What are the first four rebound heights? Fill in the following table:

Rebound	Height (ft.)
1	$A(0.6)$
2	$A(0.6)^2$
3	$A(0.6)^3$
4	$A(0.6)^4$

c. How is each term in the sequence related to the one that came before it?

Each term is 0.6 times the previous term

d. Suppose the initial height is A ft. and that each rebound, rather than being 60% of the previous height, is r times the previous height, where $0 < r < 1$. What are the first four rebound heights? What is the n^{th} rebound height?

$h_1 = Ar$

$h_2 = Ar^2$

$h_3 = Ar^3$

$h_4 = Ar^4$

$h_n = Ar^n$

e. What kind of sequence is the sequence of rebound heights?

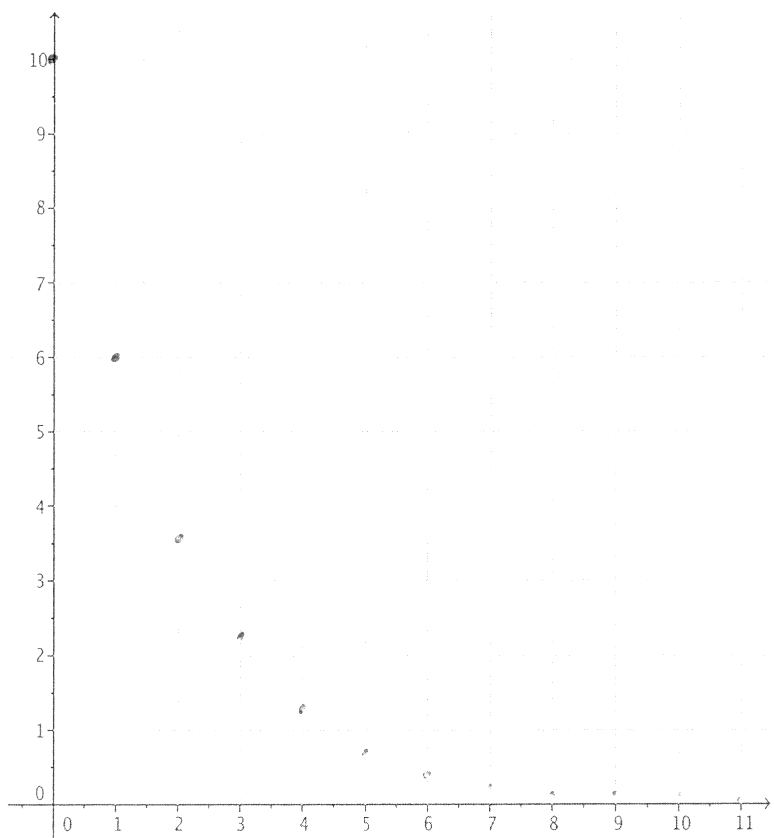
Geometric: A common ratio separates the terms $\frac{Ar^2}{Ar} = r$ $\frac{Ar^3}{Ar^2} = r$

f. Suppose that we define a function f with domain all real numbers so that $f(1)$ is the first rebound height, $f(2)$ is the second rebound height, and continuing so that $f(k)$ is the k^{th} rebound height for positive integers k. What type of function would you expect f to be?

Exponential

$f(k) = Ar^k$

g. On the coordinate plane below, sketch the height of the bouncing ball when $A = 10$ and $r = 0.60$, assuming that the highest points occur at $x = 1, 2, 3, 4, \ldots$.



h. Does the exponential function $f(x) = 10(0.60)^x$ for real numbers x model the height of the bouncing ball? Explain how you know.

No, it models the maximum height of each bounce

i. What does the function $f(n) = 10(0.60)^n$ for integers $n \geq 0$ model?

It models the max height of each bounce

j. Can n be a number OTHER THAN an integer?

Not to model the bouncing height.

Notes

Explicit formula: Tells us a specific term at an integer output/term #

Geometric sequence: $a_n = A r^{n-1}$

initial $n=1$ — common ratio — start at $n=1$

Exercises

1.

a. Jane works for a videogame development company that pays her a starting salary of $100 a day, and each day she works, she earns $100 more than the day before. How much does she earn on day 5?

$d_1 = \$100$ $d_4 = \$400$
$d_2 = \$200$
$d_3 = \$300$ $d_5 = \$500$

b. If you were to graph the growth of her salary for the first 10 days she worked, what would the graph look like? Graph her daily salary on the coordinate grid below.

Linear function: increases at a constant rate.

c. What kind of sequence is the sequence of Jane's earnings each day? Develop an explicit formula to describe the situation.

Arithmetic Sequence : has a constant rate of change:

$a_n = 100 + 100(n-1)$

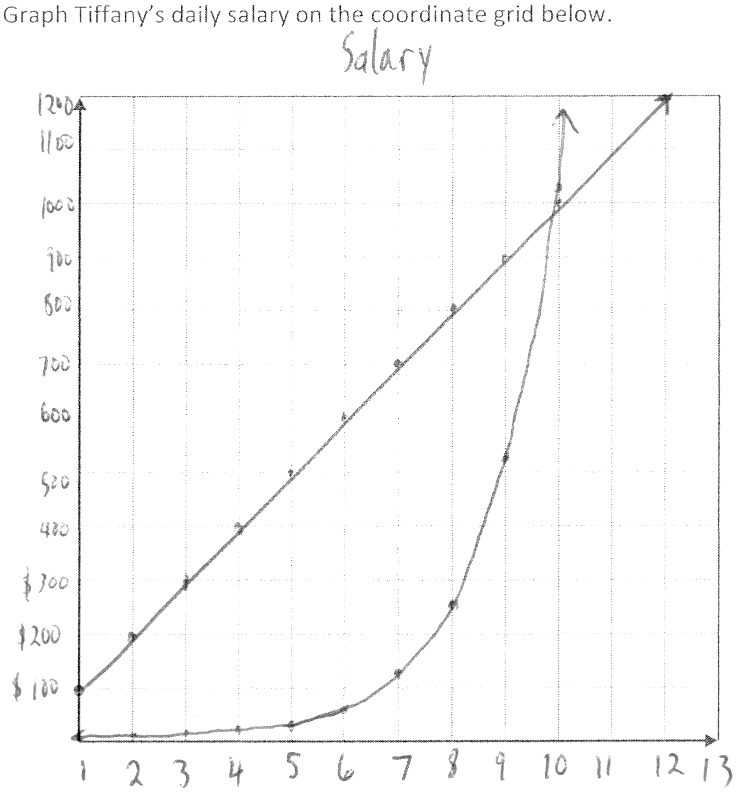
d. Tiffany works for the same videogame development company that pays her a starting salary of \$2 a day, and each day she works, she earns *twice as much* as the day before. How much does she earn on day 5?

$d_1 = 2$ $d_4 = 16$
$d_2 = 4$ $d_5 = \$32$
$d_3 = 8$

e. What kind of sequence is the sequence of Tiffany's earnings each day? Develop an explicit formula to describe the situation. Graph Tiffany's daily salary on the coordinate grid below.

Geometric: initial: 2
ratio: 2

$a_n = 2 \cdot 2^{n-1}$



What is the difference between an arithmetic and a geometric sequence?

Arithmetic: constant roc
Geometric: common ratio factor

How are the graphs of those two sequences different?

Arithmetic: linear
Geometric: exponential

Determine if the sequence is geometric. If it is, find the common ratio.

1) $-1,\ 6,\ -36,\ 216, \ldots$

$\frac{6}{-1} = -6$ $\quad \frac{216}{-36} = -6$

$\frac{-36}{6} = -6$

Yes! common ratio: -6

2) $-1,\ 1,\ 4,\ 8, \ldots$

$\frac{1}{-1} = -1$ $\quad \frac{8}{4} = 2$

$\frac{4}{1} = 4$

no common ratio

3) $4,\ 16,\ 36,\ 64, \ldots$

$\frac{16}{4} = 4$ $\quad \frac{64}{36} =$ Doesn't matter

$\frac{36}{16} = \frac{9}{4}$

no common ratio

4) $-3,\ -15,\ -75,\ -375, \ldots$

$\frac{-15}{-3} = 5$ $\quad \frac{-375}{-75} = 5$

$-75/-15 = 5$

Yes! common ratio 5

5) $-2,\ -4,\ -8,\ -16, \ldots$

$\frac{-4}{-2} = 2$ $\quad \frac{-16}{-8} = 2$

$\frac{-8}{-4} = 2$

Yes! common ratio: 2

6) $1,\ -5,\ 25,\ -125, \ldots$

$\frac{-5}{1} = -5$ $\quad \frac{-125}{25} = -5$

$\frac{25}{-5} = -5$

Yes! common ratio -5

Given the explicit formula for a geometric sequence find the first five terms and the 8th term.

7) $a_n = 3^{n-1}$

$a_8 = 3^{8-1} = 3^7 = 2187$

$a_1 = 1$

$a_2 = 3$

$a_3 = 9$

$a_4 = 27$

$a_5 = 81$

8) $a_n = 2 \cdot \left(\frac{1}{4}\right)^{n-1}$

$a_1 = 2$

$a_2 = 1/2$

$a_3 = 1/8$

$a_4 = 1/32$

$a_5 = \frac{1}{128}$

$a_8 = \frac{1}{8192}$

9) $a_n = -2.5 \cdot 4^{n-1}$

$a_1 = -2.5$ $\quad a_4 = -160$

$a_2 = -10$ $\quad a_5 = -640$

$a_3 = -40$

$a_8 = -40,960$

10) $a_n = -4 \cdot 3^{n-1}$

$a_1 = -4$ $\quad a_4 = -108$

$a_2 = -12$ $\quad a_5 = -324$

$a_3 = -36$

$a_8 = -8748$

For each of the following geometric sequences, find the common ratio. Then write the explicit formula for the sequence.

4) 10, 20, 40, 80, ...

$\times 2 \quad \times 2 \quad \times 2$

$r = 2$

$a = 10$ $\quad a_n = 10 \cdot 2^{n-1}$

5) 7, -7, 7, -7, ...

$\times(-1) \quad \times(-1) \quad \times(-1)$

$r = -1$

$a = 7$ $\quad a_n = 7(-1)^{n-1}$

6) 3, -12, 48, -192, ...

$\times(-4) \quad (-4) \quad -4$

$r = -4$

$a = 3$ $\quad a_n = 3(-4)^{n-1}$

7) 162, 108, 72, 48, ...

$\frac{108}{162} = \frac{2}{3} = r$

$a = 162$ $\quad a_n = 162\left(\frac{2}{3}\right)^{n-1}$

8) 100, 50, 25, 12.5, ...

$\div 2 \quad \div 2$

$r = 1/2$

$a = 100$

$a_n = 100\left(\frac{1}{2}\right)^{n-1}$

Given the first term and the common ratio of a geometric sequence find the first five terms and the explicit formula.

15) $a_1 = 0.8,\ r = -5$

$a_n = 0.8(-5)^{n-1}$

$a_1 = 0.8$ $a_2 = -4$ $a_3 = 20$ $a_4 = -100$ $a_5 = 500$

16) $a_1 = 1,\ r = 2$

$a_n = 1 \cdot 2^{n-1}$

$a_1 = 1$ $a_2 = 2$ $a_3 = 4$ $a_4 = 8$ $a_5 = 16$

17) $a_1 = -4,\ r = 6$

$a_n = -4(6)^{n-1}$

$a_1 = -4$ $a_2 = -24$ $a_3 = -144$ $a_4 = -864$ $a_5 = -5184$

18) $a_1 = 4,\ r = 6$

$a_n = 4(6)^{n-1}$

$a_1 = 4$ $a_2 = 24$ $a_3 = 144$ $a_4 = 864$ $a_5 = 5184$

19) $a_1 = 2,\ r = 6$

$a_n = 2(6)^{n-1}$

$a_1 = 2$ $a_2 = 12$ $a_3 = 72$ $a_4 = 432$ $a_5 = 2592$

20) $a_1 = -4,\ r = 4$

$a_n = -4(4)^{n-1}$

$a_1 = -4$ $a_2 = -16$ $a_3 = -64$ $a_4 = -256$ $a_5 = -1024$

Given two terms in a geometric sequence find the 8th term and the ~~recursive formula~~.

23) $a_4 = -12$ and $a_3 = -6$

$\frac{-12}{-6} = 2$ $r = 2$

$a_4 = -12$ $a_5 = -24$ $a_6 = -48$ $a_7 = -96$ $a_8 = -192$

24) $a_5 = 768$ and $a_2 = 12$

$a_2 = 12$ $a_5 = 12 \cdot r^3$

$768 = 12 \cdot r^3$

$r = 4$

$a_5 = 768$ $a_6 = 3072$ $a_7 = 12,288$ $a_8 = 49,152$

25) $a_1 = -2$ and $a_5 = -512$

$a_1 = -2$ $a_5 = -2r^4$

$-512 = -2r^4$

$r = 4$

$a_5 = -512$ $a_6 = -2048$ $a_7 = -8192$ $a_8 = -32768$

26) $a_5 = 3888$ and $a_3 = 108$

$3888 = 108r^2$

$r = 6$

$a_5 = 3888$ $a_6 = 23,328$ $a_7 = 139,968$ $a_8 = 839,808$

Gabe and Erik are finding the 9th term of the geometric sequence -5, 10, -20, ...

Is either of them correct? Explain.

Gabe

$r = \frac{10}{-5} = -2$

$a_9 = -5(-2)^{9-1}$

$= -5(512)$

$= -2560$

NO

$-5(-2)^{9-1}$

$-5(-2)^8$

$-5(256)$

Erik

$r = \frac{10}{-5} = -2$

$a_9 = -5(-2)^{9-1}$

$= -5(-256)$

$= 1280$

In a geometric sequence in which all of the terms are positive, the second term is 2 and the fourth term is 10. What is the value of the seventh term in the sequence?

A. $10\sqrt{5}$
B. 20
C. ~~50~~
D. $50\sqrt{5}$
E. ~~250~~

$a_2=2$
$a_4=10$
$a_4=2r^2$
$10=2r^2$
$r=\sqrt{5}$

$a_4=10$
$a_5=10\sqrt{5}$
$a_6=50$
$a_7=50\sqrt{5}$

60. The sum of an infinite geometric series with first term a and common ratio $r < 1$ is given by $\frac{a}{1-r}$. The sum of a given infinite geometric series is 200, and the common ratio is 0.15. What is the second term of this series?

F. 25.5
G. 30
H. 169.85
J. 170
K. 199.85

$200=\frac{a}{1-0.15}=200=\frac{a}{0.85}$

$200(0.85)=a$

$a_1=170$

$a_2=170(0.15)=25.5$

Find the missing term of the geometric sequence.

45, ___, 1620, ...

a. 9720 b. 51 c. 6 d. 270

$1620=45r^2$

$r=6$

$$\frac{d}{dx} f^{-1}(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{f'(3)}$$

Inverses, switch x-y

$$f^{-1}(5) = 3$$

$$\frac{d}{dx} f^{-1}(4)$$

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

$$\frac{d}{dx} \ln(x^3) = \boxed{\frac{1}{x^3} \cdot 3x^2} = \frac{3x^2}{x^3} = \frac{3}{x}$$